Math2050A Term1 2017
Tutorial 8 and 9, Nov 9 and Nov 23

Tutorial 8:

In examples

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x^2+1}$, we have stated that if $f : \mathbb{R} \to \mathbb{R}$ is continuous such that both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exists, then f is uniformly continuous. For this example, one can check directly that f is Lipschitz.

(b) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$, we know that $f \restriction_{[1,\infty)}$ is Lipschitz and hence uniformly continuous. One can show that f is uniformly continuous together with the fact that $f \restriction_{[0,2]}$ is uniformly continuous.

Tutorial 9: (Exercises)

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous with $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$. Show that f attains min or max.

2. Suppose $f : [0, 1] \to [0, 1]$ is continuous. Show that $\exists c \in [0, 1]$ such that $f(c) = c$.

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous periodic function with period p. Show that f is uniformly continuous.
   Definition: We say that f is periodic with period p if $p > 0$ and $f(x + p) = f(x)$ for every $x \in \mathbb{R}$

4. This is from exam paper (2014-2015).

   For each integer $n \geq 2$, define $f_n : [\frac{1}{2}, 1] \to \mathbb{R}$ by $f_n(x) = x^n + x$. Note f_n is monotone continuous for each $n \geq 2$.

   (a) Show that for each n, there is a unique $z_n \in [\frac{1}{2}, 1]$ with $f_n(z_n) = 1$.

(b) Show that $\lim_{n\to\infty} z_n$ exists in $\mathbb{R}$. Can you find the limit? Why?

For 4(b), Let $\beta := \lim_{n\to\infty} z_n$. Note if $0 < \beta < 1$, then $\lim_{n\to\infty} z_n^n = 0$. This is root test.